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DK paper ..

"free categories": one way of building a free cat $C \mapsto FC$ is

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, "u"] \arrow[rr, bend right, "F"'] & \mathrm{Graphs} \arrow[r, "\mathrm{Free}"] & \mathrm{Cat} \end{tikzcd}$$

explicitly FC has a generator Fc $\forall\, c \in \mathrm{Hom}(C)$.

any map in FC can be written as a finite composition of generators

The standard resolution is a simplicial set F_*C

$F_kC \in O\text{-Cat}$

$sO\text{-Cat}$:

$\mathrm{ob} = \{ X : \Delta^{op} \to O\text{-Cat} \}$

$$\begin{tikzcd} {[n]} \arrow[r, mapsto] \arrow[d] & X_n \\ {[m]} \arrow[r, mapsto] & X_m \arrow[u] \end{tikzcd}$$

There are 2 different things called "simplicial cats"

(1) in the sense above, as a simplicial object in Cat.

?? ie. a functor $X : \Delta^{op} \to \mathrm{Cat}$.

(2) a category enriched in sSets.

ie. a cat C – ob = ...

$\mathrm{Hom}_C(x,y)_\bullet \in \mathrm{sSet}$.

this is one of our models for ∞-cats.

To DK, an object in $sO\text{-Cat}$ is a simplicial object in $O\text{-Cat}$

ie. a functor $\Delta^{op} \to O\text{-Cat}$

$[n] \mapsto X_n \in$

---

what we want

$$\begin{tikzcd} (\text{model cats}) \arrow[r, "L^H"] \arrow[rr, bend left] & \mathrm{sCat} \arrow[r, leftrightsquigarrow] & \infty\text{-Cat} \end{tikzcd}$$

sCat: simp. enriched cats

$$\begin{tikzcd} M \arrow[r, mapsto] \arrow[d, mapsto] & L^H M \in \mathrm{sCat} \\ L^{DK} M \arrow[ur, squiggly, "?"] & \\ sO\text{-Cat} \arrow[u] & \end{tikzcd}$$

$sO\text{-Cat}$: simp. obj. in O-Cat

$sO\text{-Cat} \longrightarrow \mathrm{sCat}$

A sCat (simp. enriched cat) is an sO-Cat (simp. ob. in Cat) whose face/degeneracy maps are bijective on objects.

(nLab page on simplicial cats)

Q: Let $(\mathcal{C}, W)$ be a cat. w/ weak eq's.

Then $F_*\mathcal{C}$ & F_*W are $s\mathcal{O}$-Cats $\quad \mathcal{O} = ob(\mathcal{C})$.

Given an $s\mathcal{O}$-Cat A & a sub $s\mathcal{O}$-cat $A_0 \subseteq A$

what is $A[A_0^{-1}]$?
$\uparrow$
$s\mathcal{O}$-Cat.

$A[A_0^{-1}] : \Delta^{op} \longrightarrow \mathcal{O}\text{-Cat}$

$[n] \longmapsto A[A_0^{-1}]_n \overset{?}{=} A_n[A_{0,n}^{-1}]$

ordinary loc'n of cats.

both $\mathcal{O}$-Cats. & $A_{0,n} \overset{?}{\subseteq} A_n$

$\{A_{0,n} = A_0[n]\}$

For us, $F_*\mathcal{C}[F_*W^{-1}] \; - (n) = F_n\mathcal{C}[F_nW^{-1}]$
$\quad ||$
$L(\mathcal{C},W) = L\mathcal{C}$

x

claim $\pi_0 L\mathcal{C} = \mathcal{C}[W^{-1}]$

what is π_0 of an $s\mathcal{O}$-Cat?

$\pi_0 : s\mathcal{O}\text{-Cat} \longrightarrow \text{Cat}$

$X \longmapsto \pi_0 X$ — $ob = \mathcal{O}$

Hom$_{\pi_0 X}$

Pick $x, y \in \mathcal{C}$.
$L\mathcal{C} \in s\mathcal{O}\text{-Cat}$
$= \{L\mathcal{C}_n\}_{n\in\mathbb{N}}$ $\quad L\mathcal{C}_0, L\mathcal{C}_1 \in \mathcal{O}\text{-Cat}$
$\uparrow$
$\mathcal{O}$-Cat.

"components of $L\mathcal{C}$" are

$\mathrm{Hom}_{L\mathcal{C}_0}(x,y) / \sim$ induced from $\mathrm{Hom}_{L\mathcal{C}_1}(x,y)$

$L\mathcal{C}_0 = F\mathcal{C}[FW^{-1}]$

$L\mathcal{C}_1 = F^2\mathcal{C}[F^2W^{-1}]$

Pick $x, y \in \mathcal{C}$. Then $\mathrm{Hom}_{\pi_0 L\mathcal{C}}(x,y) = \mathrm{Hom}_{L\mathcal{C}_0}(x,y) /$ $f \sim g$ iff there's $(\cdot \dashrightarrow)$ in $L\mathcal{C}_1$

Cf. π_0 of sSets.
Let $X \in$ sSet. Then
$\pi_0 X$ is a set
$\pi_0 X = X_0 / (x \sim x'$ iff there's an edge $x \to x'$ or $x' \to x$ in $X_1)$

Q: Where are homotopies in $L\mathcal{C}$?

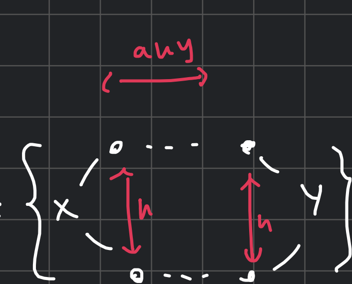
1. Describe $L\mathcal{C}_0 = F\mathcal{C}[FW^{-1}]$ — relation to $F\mathcal{C}[W^{-1}]$?

2. Look at $L\mathcal{C}_1$ ...

0. Look up in May [8], Bousfield-Kan [3, Ch. VIII] what π_0 of a simp. obj. in Cat means.

3. Look at the next paper & compare Hammock loc'n w/ this one.

$\mathcal{C} \longmapsto L^H\mathcal{C}$ — $ob: ob(\mathcal{C})$

$L^H\mathcal{C}(x,y)_\bullet$ — $(n): L^H\mathcal{C}(x,y)_n = \Big\{$  $\Big\}$.

nerve of a sO-Cat ...

$X = \{X_n\}_{n \in \mathbb{N}}$

$\uparrow$ O-cat.

Taking $\tilde{N}X : \Delta^{op} \times \Delta^{op} \longrightarrow Set$

$(i,j) \longmapsto N(X_i)_j = \{\text{strings } (x_0 \to x_1 \to \cdots \to x_j) \text{ in } X_i\}$.

Then define $NX = \operatorname{diag}(\tilde{N}X)$

$NX_i = N(X_i)_i = \{\text{strings } (x_0 \to x_1 \to \cdots \to x_i) \text{ in } X_i\}$.